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Approximation of conjugate Fourier series of a function belonging to $Lip(\alpha)$ class using $(\overline{N}, p_n, q_n)(E, s)$ - summability method

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Abstract

Degree of approximation of functions of different classes has been studied by several researchers by using different summability methods. In the proposed paper a new theorem has been established for the approximation of a function belonging to the $Lip(\alpha)$-class by $(\overline{N}, p_n, q_n)(E, s)$ **-** product summability means of a conjugate series of Fourier series. The result obtained here is a generalization of several known theorems.

Key Words: Degree of approximation, conjugate Fourier series, $Lip(\alpha)$ - class, $(\overline{N}, p_n, q_n)(E, s)$ - mean, Lebesgue integral.

AMS Subject classification (2010): Primary 42A24; Secondary 41A25, 42B025, 42B08, 40C05.

1 Introduction

Approximations of a periodic function in a given period arose in connection with convergence of Fourier series. The objective of such study is to estimate minimum error obtained out of the approximations of a function in a given interval. The most important trigonometric polynomials used in approximation theory are obtained by different linear summability methods of Fourier series of 2π periodic functions in the real line R (for example, Cesaro mean, Norlund mean, Matrix mean etc.). Many advances in the theory of trigonometric approximation have been studied by different researchers for periodic function of Lipschitz classes. The degree of approximation of functions belonging to classes $Lip(\alpha)$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$, $(r \geq 1)$ through trigonometric Fourier series expansion by using different summability methods has been proved by various investigators like [3], [4], [5], [7], [8], [9] and [10]. Furthermore, the degree of approximation of functions belonging to $Lip(\alpha)$ - class through conjugate series of the Fourier series expansion of a function by using $(E,1)(C,1)$ mean has been proved by Nigam *et al.* (see, 6). In an attempt to have an advance study in this direction, here we have established a new theorem on $(\overline{N}, p_n, q_n)(E, s)$ mean of conjugate series of the Fourier series.

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2 Definitions and Notations

Let $\sum u_n$ be a given infinite series with the sequence of partial sum (s_n). Let (p_n) and (q_n) be sequences of positive real number such that,

$$P_n=\sum_{k=0}^{n} P_k \text{ and } Q_n=\sum_{k=0}^{n} q_k.$$

The sequence to sequence transformation

$$t_n^{\bar{N}} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_k s_k, \tag{2.1}$$

where

$$R_n = p_0 q_n + p_1 q_{n-1} + \cdots\cdots + p_n q_0 (\neq 0) \quad (p_{-1} = q_{-1} = R_{-1} = 0).$$

Defines $(\bar{N}, p_n, q_n)$ mean of the sequence (s_n) generated by the sequence of coefficient (p_n) and (q_n)
and it is denoted by $t_n{}^{\bar{N}}$.
If $\lim_{n\to\infty} t_n{}^{\bar{N}} \to s$, then the series $\sum u_n$ *is* $(\bar{N}, p_n, q_n)$ summable to s.

The necessary and sufficient conditions for regularity of $(\bar{N}, p_n, q_n)$ summability are

(i) $\frac{p_n q_n}{R_n} \to o$ for each integer $k \geq o$ as $n \to \infty$ and

(ii) $|\sum_{k=0}^{n} p_k q_k| < C|R_n|$, where c is any positive integer of n.

The sequence to sequence transformation

$$E_n^s = \frac{1}{(1+s)^n}\sum_{v=0}^{n} \binom{n}{s} s^{n-v} s_v. \tag{2.2}$$

Defines the (E, s) mean of the sequence (s_n) and it is denoted by (E_n^s).

If $E_n^s \to s$ as $n \to \infty$, then $\sum u_n$ is summable to s with respect to (E, s) summability and (E, s) method is regular (see [1]).
Now, we define a new composite transformation $(\bar{N}, p_n, q_n)$ over (E, s) of (s_n) as

$$T_n^{\bar{N}E} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_k \ (E_k{}^s) = \frac{1}{R_n}\left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{s} s^{k-v} s_v\right\}. \tag{2.3}$$

If $T_n^{\bar{N}E} \to \infty$ as $n \to \infty$, then $\sum u_n$ is summable to s by $(\bar{N}, p_n, q_n)$ (E, s) summability method. Moreover, as $(\bar{N}, p_n, q_n)$ and (E, s) methods are both regular, so $(\bar{N}, p_n, q_n)$ (E, s) method is also regular.
Thus, we may write,

$$T_n^{\bar{N}E} = t_n^{\bar{N}}\{E_n^s \ (s_n)\} \to s \ (n \to \infty).$$

Remark 1. If we put $q_n = 1$, in equation (2.1) then $(\bar{N}, p_n, q_n)$ summability method reduces to $(\bar{N}, p_n)$ summability and for $p_n = 1$, the $(\bar{N}, p_n)$ summability reduces to $(C, 1)$ summability.

Let (f) be 2π periodic function belonging to $L^r\ [0, 2\pi]$ $(r \geq 1)$ with the partial sum $s_n\ (f)$ defined by,

$$s_n\ (f) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k coskx + b_k\ sinkx\ ). \tag{2.4}$$

The conjugate series of the Fourier series (2.4) is given by

$$\bar{f} = \sum_{k=1}^{\infty}(a_k coskx - b_k\ sinkx\ ) \quad (2.5).$$

A function $f\ \epsilon\ Lip\ (\alpha),\ if$

$$|f(x+t) - f(x)| = O(|t^{\alpha}|) \quad (0 < \alpha \leq 1)(t > 0).$$

The L_∞ - norm of a function $f: R \to R$ is defined by

$$\|f\|_\infty = sup\{|f(x)|: x\epsilon R\}$$

and L_r - norm of a function $f: R \to R$ is defined by

$$\|f\|_r = \left(\int_{[0,\pi]} |f(x)|^r dx\right)^{\frac{1}{r}} \quad (r \geq 1).$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial (t_n) of order n under $\|.\|_\infty$ is defined by

$$\|t_n - f(x)\|_\infty = sup\{|t_n(x) - f(x): x\epsilon R|\}$$

and the degree of approximation of $E_n(f)$of a function $f\ \epsilon\ L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r .$$

We use the following functions throughout the paper,

$$\varphi(t) = f(x+t) + f(x-t)$$

$$\overline{K_n}(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} \frac{\cos\frac{t}{2} - \cos\left(v+\frac{1}{2}\right)t}{\sin\frac{1}{2}} \right\}.$$

3 Known Theorem

Theorem1. [6] If a function $\overline{f}$, conjugate 2π periodic function f, belonging to $Lip(\alpha)$ class, then its degree of approximation $(E,1)(C,1)$ means of conjugate Fourier series is given by

$$sup\,|(\overline{EC})_n^1(x) - f| = O\{(n+1)^{-\propto}\} \qquad (0 < \alpha < 1),$$

(3.1)

where $(\overline{EC})_n^1$ denotes the $(E,1)(C,1)$ transform as define in (2.3).

4 Main Theorem

The objective of this paper is to prove the following theorem

Theorem 2. If a function $\overline{f}$, conjugate 2π periodic function f, belonging to $Lip(\alpha)$ class, then its degree of approximation by $(\overline{N}, p_n, q_n)$ (E,q) means of conjugate series of the Fourier series is given by

$$\left\|\overline{T}_n^{\overline{N}E} - \overline{f}\right\|_\infty = \mathrm{O}\left[\frac{1}{(n+1)^\alpha}\right] \qquad (0 < \alpha < 1),$$

(4.1)

where $T_n{}^{\overline{N}E}$ denotes the $(\overline{N}, p_n, q_n)$ (E,s) transform as defined in (2.3).

To prove the theorem we need the following Lemmas.

Lemma 1. $|\overline{K_n}(t)| = o(n) \quad \left(0 \le t \le \frac{1}{n+1}\right).$

Proof. For $0 \le t \le \frac{1}{n+1}$, we have $\sin nt \le n \sin t$

$$|\overline{K_n}(t)| = \frac{1}{2\pi R_n} \left| \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} \frac{\cos\frac{t}{2} - \cos\left(v+\frac{1}{2}\right)t}{\sin\frac{1}{2}} \right\} \right|$$

$$= \frac{1}{2\pi R_n} \left| \sum_{k=0}^{n} p_k q_k \left\{ \frac{1}{(1+s)^k} \sum_{v=0}^{k} \binom{k}{v} s^{k-v} \left(\frac{\cos\frac{t}{2} - \cos vt . \cos\frac{t}{2} + \sin vt \, \sin\frac{t}{2}}{\sin\frac{1}{2}} \right) \right\} \right|$$

$$=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\frac{\cos\frac{t}{2}\left(2sin^2 v\frac{t}{2}\right)}{\sin\frac{1}{2}}+\sin vt\right)\right\}\right|$$

$$=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(o\left(2\sin v\frac{t}{2}\sin v\frac{t}{2}\right)\right)+\right.\right.$$

$v\sin t$

$$=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}(o(v)+o(v))\right\}\right|$$

$$=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}o(k)\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\right\}\right|$$

$$=\frac{O(n)}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}(1+s)^k\right\}\right|$$

$=O(n)$.

Lemma 2. $|\overline{K_n}(t)| = O\left(\frac{1}{t}\right) \quad \left(\frac{1}{n+1} < t \le \pi\right).$

Proof For $\left(\frac{1}{n+1} < t \le \pi\right)$ and by using Jordans Lemma, $\sin\frac{t}{2} \ge \frac{t}{\pi}$, and $\sin nt \le 1$.

$$|\overline{K_n}(t)| = \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\frac{\cos\frac{t}{2}-\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{1}{2}}\right\}\right|$$

$$=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\frac{\cos\frac{t}{2}-\cos vt.\cos\frac{t}{2}+\sin vt\sin\frac{t}{2}}{\sin\frac{1}{2}}\right)\right\}\right|$$

$$=\frac{1}{2\pi R_n}\left(\frac{\pi}{t}\right)\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\cos\frac{t}{2}\left(2sin^2 v\frac{t}{2}\right)+\right.\right.\right.$$

$\sin vt$

$$=\frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\right\}\right|$$

$=O\left(\frac{1}{t}\right)$.

5 Proof of main theorem

Let $\bar{s_n}(f)$ denotes the nth partial sum of the series (2.5), then we have (see [2])

$$\bar{s}_n(x)-\bar{f}(x)=\frac{1}{2\pi}\int_{[0,\pi]}\varphi(t)\left\{\frac{\cos\frac{t}{2}-\cos\left(v+\frac{1}{2}\right)t}{\sin\frac{1}{2}}\right\}dt.$$

So, using the and (E,s) transform of $\bar{s_n}(f)$, we have

$$\bar{E}_n^s-\bar{f}(x)=\frac{1}{2\pi(1+s)^n}\int_{[0,\pi]}\frac{\varphi(t)}{\sin\frac{1}{2}}\left\{\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\left(\cos\frac{t}{2}-\cos\left(k+\frac{1}{2}\right)t\right)\right\}dt.$$

Furthermore, considering the $(\bar{N},p_n,q_n)$ (E,s) transform of $\bar{s_n}(f)$ by $\bar{T}_n^{\bar{N}E}$, we have

$$\bar{T}_n^{\bar{N}E} - \bar{f}(x) = \frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_k \int_{[0,\pi]} \frac{\varphi(t)}{\sin\frac{1}{2}}\left\{\sum_{v=0}^{k}\binom{k}{v} s^{k-v}\left(\cos\frac{t}{2} - \cos\left(v+\frac{1}{2}\right)t\right)\right\}dt$$

$$= \left[\int_{\left[0,\frac{1}{n+1}\right]} + \int_{\left[\frac{1}{n+1},0\right]}\right]\varphi(t)\bar{K}_n(t)dt$$

$$= I_1 + I_2 (\text{say}).$$

Now using Lemma-1,

$$I_1 = \int_0^{\frac{1}{n+1}} |\varphi(t)|\,|\overline{K_n}(t)|\,dt$$

$$= o(n)\int_0^{\frac{1}{n+1}} |\varphi(t)| dt$$

$$= o(n)\int_0^{\frac{1}{n+1}} |t|^{\alpha} dt$$

$$= o(n)\left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{\alpha+1}}$$

$$= \frac{O(n)}{\alpha+1}\left[\frac{1}{(n+1)^{\alpha+1}}\right]$$

$$= O\left[\frac{1}{(n+1)^{\alpha+1}}\right] \qquad (5.1)$$

Next, by using Lemma – 2

$$|I_2| = \int_{\frac{1}{n+1}}^{\pi} |\varphi(t)|\,K_n|(t)|\,dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} |t^{\alpha}|\ O\left(\frac{1}{t}\right) dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1}\ dt$$

$$= \left[\frac{t^{\alpha}}{\alpha}\right]_{\frac{1}{n+1}}^{\pi}$$

$$= O\left[\frac{1}{(n+1)^{\alpha}}\right] \qquad (5.2)$$

Cleary, by using (5.1) and (5.2), we have

$$\left|\bar{T}_n^{\bar{N}E} - \bar{f}\right| = O\left[\frac{1}{(n+1)^{\alpha}}\right]$$

Hence, if $f \in Lip(\alpha)$ and $0 < \alpha < 1$, then

$$\left\|\bar{T}_n^{\bar{N}E} - \bar{f}\right\|_{\infty} = \frac{Sup}{-\pi<x<\pi}\left|\bar{T}_n^{\bar{N}E} - \bar{f}\right| = O\left[\frac{1}{(n+1)^{\alpha}}\right].$$

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